

BLACK HOLE INITIAL DATA WITH A HORIZON OF PRESCRIBED GEOMETRY

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ABSTRACT. The purpose of this work is to construct asymptotically flat, time symmetric initial data with an apparent horizon of prescribed intrinsic geometry. To do this, we use the parabolic partial differential equation for prescribing scalar curvature. In this equation the horizon geometry is contained within the freely specifiable part of the metric. This contrasts with the conformal method in which the geometry of the horizon can only be specified up to a conformal factor.

1. INTRODUCTION

In this work, the excision approach to constructing black hole initial data is used. Specifically, black hole initial data for the Einstein equations is taken to consist of the following: (1) a manifold $M = \mathbb{R}^3 \setminus K$, where K is a union of a finite number of balls. (2) a metric g on M asymptotic to the flat background metric δ ; in this work we take this in the very strong sense that there exist constants C_j for all $j \geq 0$ such that $|r^{j+1}\partial_j(g_{kl} - \delta_{kl})| \leq C_j$, where r is the coordinate radius. (3) a second rank tensor field k asymptotic to 0. (4) a vectorfield J asymptotic to 0. (5) a scalar field ρ asymptotic to 0. Furthermore, we require that (g, J, k, ρ) satisfy the Einstein constraint equations

$$\begin{aligned} R(g) + (\text{tr}k)^2 - |k|_g^2 &= 16\pi\rho \\ \nabla \cdot k - \nabla(\text{tr}k)^2 &= -8\pi J, \end{aligned}$$

the dominant energy condition

$$\rho > |J|_g,$$

and, in addition, an apparent horizon boundary condition on ∂K . To write the latter, let n, H be the outward unit normal and mean curvature of ∂K , respectively. The apparent horizon boundary condition is that on ∂K , one has

$$\begin{aligned} \theta^+ &\equiv H + k(n, n) - \text{tr}k = 0 \\ \theta^- &\equiv -H + k(n, n) - \text{tr}k \leq 0. \end{aligned}$$

In addition, there should be no other compact surfaces satisfying $\theta^+ \leq 0$.

The interpretation of the data is that M is a spacelike slice with induced metric g and second fundamental form k in an asymptotically flat spacetime. The fields J and ρ are, respectively, the local momentum current density and mass energy density according to an observer comoving with M . The conditions that we have assumed on ∂K imply, assuming weak or strong cosmic censorship and the weak or

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strong energy condition, that if the data were extended inside of ∂K , this surface would form the boundary of the *totally trapped region* (for a definition see [17]), and thus ∂K is the *apparent horizon*. The surface ∂K represents a black hole within M since, again, if (M, g, k, J, ρ) were extended inside ∂K , this set would represent the outer boundary of the largest set on M which, based solely on the geometry of (M, g, k, J, ρ) , must be contained within the black hole region of space time [9], [17]. Thus, although ∂K is, in general, not the set in which the spacelike slice intersects the event horizon, within the initial data set itself it provides the best possible representative of the black hole. In addition, if the censorship and energy assumptions above hold, any data inside ∂K will not affect the Cauchy development outside of the event horizon, and for this reason we exclude it from discussion. Finally, as shown by Hawking [8], ∂K will always consist of a finite union of topological spheres, and so there has been no loss of generality in the topological assumptions on M .

The problem of the construction of black hole initial data with apparent horizon boundaries has received much attention in recent years. It would be impossible to cite every relevant work here, but the interested reader may see, for instance [4], [5], [6], [7], [11]. In most of these works, the main tool for constructing black hole initial data with an apparent horizon boundary has been the conformal method, and considerable progress has been made. This method involves taking an initial “free” metric on $\mathbb{R}^3 \setminus K$ and modifying it with a conformal factor to ensure that the constraint equations are satisfied. The apparent horizon condition appears as a Neumann boundary condition on ∂K . However, with this method one has no control over the values of the conformal factor on the horizon; it is therefore impossible to specify the horizon geometry with this method.

Since in the present work we want to prescribe the geometry of the apparent horizon, we use instead the parabolic equation for prescribing scalar curvature, for which the horizon geometry will be contained in the freely specifiable part of the metric. In [1], Bartnik does this for quasi-spherical metrics, which allows for the construction of apparent horizons that are intrinsically round. In the present work we relax this condition. Since this is a first attempt in that direction we are also going to make the simplifying assumptions that the horizon only has one component, the initial data set has moment of time symmetry $k = 0$, and the local mass density ρ is compactly supported. This reduces the problem to the following: *Given a compactly supported local mass density $\rho \geq 0$ on $\mathbb{R}^3 \setminus B_{r_0}(0)$, and a metric h on $\partial B_{r_0}(0) \equiv \mathbb{S}^2$, construct a metric g on $\mathbb{R}^3 \setminus B_{r_0}(0)$, asymptotic to the standard flat metric δ_{ij} , satisfying $R(g) = 16\pi\rho$, and such that $g|_{\partial B_{r_0}(0)} = h$.*

To attack this problem we make the standard spherical-polar identification of M with $[r_0, \infty) \times \mathbb{S}^2$ and construct g in the form

$$(1) \quad g = u^2 dr^2 + r^2 \gamma,$$

where γ vanishes on ∂_r and $g|_{\mathbb{S}^2} = r^2 \gamma$. The parabolic scalar curvature equation, which we write in the next section, may be viewed as a second order parabolic partial differential equation for the radial component u of the metric as long as $\gamma^{AB} r \partial_r \gamma_{AB} > -4$. The component γ appears in the coefficients along with the scalar curvature R .

To see how we can obtain the apparent horizon condition for a metric written in the form (1) note that the Schwarzschild metric cast into this form reads

$$g_s = \frac{1}{1 - \frac{r_0}{r}} dr^2 + r^2 \gamma,$$

where γ is a fixed round metric on \mathbb{S}^2 . Of course, at first glance it appears that the metric is singular at r_0 , but it is well known that an appropriate coordinate transformation shows that g_s can actually be extended to the boundary as a smooth metric on a manifold with boundary. The blow-up in the normal component is, in fact, responsible for the surface $r = r_0$ being an apparent horizon for this metric. Indeed, this follows since the extrinsic curvature of the foliation spheres of a metric in the form (1) is

$$\chi = \frac{r}{u} \left(\gamma + \frac{1}{2} r \frac{\partial \gamma}{\partial r} \right),$$

and so for the Schwarzschild metric one has $\chi = \sqrt{r} \sqrt{r - r_0} \gamma$. Of course, in this case we actually know that S_{r_0} is a spacelike slice of the event horizon.

Our approach to obtaining a horizon at $r = r_0$ will be to imitate the Schwarzschild case as closely as possible while striving for the greatest generality in the constructed metric. Namely, we shall construct metrics in the form

$$g_s = \frac{v^2}{1 - \frac{r_0}{r}} dr^2 + r^2 \gamma$$

by choosing γ at the outset and using the parabolic scalar curvature equation to solve for v in order to get $R(g) = 16\pi\rho \geq 0$. Then the horizon geometry is just given by $r^2\gamma$ and is thus contained within the freely specifiable part of the metric. We may implement this procedure for a large class of γ . In particular, one has

Main Theorem. *Let h be a metric on \mathbb{S}^2 with positive Gauss curvature. Let $\varepsilon > 0$ and let $\bar{\gamma}$ denote the fixed round metric on \mathbb{S}^2 . Let $\gamma(r)$ be a smooth family of metrics on $[r_0, \infty)$ that satisfies, for some large $r_2 > r_0 + \varepsilon$, the following:*

$$\begin{aligned} r_0^2 \gamma(r_0) &= h \\ \gamma(r) &\equiv \gamma(r_0), \quad r \in [r_0, r_0 + \varepsilon) \\ \gamma(r) &\equiv \bar{\gamma}, \quad r > r_2 \\ r \frac{\partial \gamma_{AB}}{\partial r} \gamma^{AB} &> -4 \\ \kappa(\gamma) &> 0 \end{aligned}$$

Then for a compactly supported function ρ on $M = [r_0, \infty) \times \mathbb{S}^2$ satisfying $16\pi\rho < \kappa$, and which also vanishes on $[r_0, r_0 + \varepsilon)$, there exists asymptotically flat time symmetric initial data $(g, 0)$ on M of mass density ρ such that g has the form

$$g = \frac{v^2}{1 - \frac{r_0}{r}} dr^2 + r^2 \gamma,$$

where v is C^∞ , independent of r on $[r_0, r_0 + \varepsilon)$, and bounded above and below by positive constants. The metric g has an apparent horizon at $r = r_0$ with intrinsic geometry (\mathbb{S}^2, h) ; the horizon is, in fact, totally geodesic.

This theorem is an immediate corollary of a slight generalization to be proved at the end of the last section, in which we replace the conditions $\kappa > 0, 16\pi\rho < \kappa$ with a more general, but more complicated, integral condition involving $\rho, \bar{H}, A, \kappa, \kappa|_{r_0}$.

A few remarks are also in order concerning the other conditions on the free part of the metric γ . The first of these is obviously what allows us to specify the horizon geometry. The second is a technical condition, but only affects the generality of γ on the region $[r_0, r_0 + \varepsilon]$, which can be made arbitrarily small; we shall refer to this region $[r_0, r_0 + \varepsilon] \times \mathbb{S}^2$ as the *collar region*. The third is a technical restriction that can be removed and replaced with an assumption on the decay of γ to $\bar{\gamma}$. However, doing so would require a detailed analysis of the asymptotic behavior, and this would detract from the purpose of the current work. The next to last condition serves the technical purpose that it causes the parabolic scalar curvature equation to be uniformly parabolic for r strictly greater than r_0 , but it also implies that the mean curvature of the foliation will be positive since the mean curvature of these spheres can, in general, be calculated to be

$$H = \frac{1}{ru} \left(2 + \frac{1}{2} r \frac{\partial \gamma_{AB}}{\partial r} \gamma^{AB} \right).$$

This fact ensures that outside of $r = r_0$ there are no other compact surfaces satisfying $\theta^+ \leq 0$, and so $r = r_0$ is an apparent horizon.

The outline of the paper is as follows: In the next section we write the parabolic scalar curvature equation and present some necessary results concerning this equation that were obtained in previous works. In the last section we prove the slight extension of our main theorem in two steps. In the first step we construct the metric on the collar region by solving an elliptic equation for v that we obtain from the parabolic scalar curvature equation by separation of variables; the conditions on the collar region are precisely what allows us to do this. This step is used to overcome the fact that the parabolicity of the parabolic scalar curvature equation must break down on the horizon. In the second step, which is contained in the proof of Theorem 6, the metric exterior to the collar region is constructed using the results from the first section.

2. THE PARABOLIC SCALAR CURVATURE EQUATION

The proof of the main theorem is essentially a problem in prescribed scalar curvature. That is, given the “free” part of the metric γ , and the function ρ , we want to choose the remaining component u such that the metric $g = u^2 dr^2 + r^2 \gamma$ has scalar curvature $R = 16\pi\rho$. To do this, we use the parabolic scalar curvature equation, which relates u to R, γ :

$$(2) \quad \bar{H} r \frac{\partial u}{\partial r} = u^2 \Delta_\gamma u + Au - \left(\kappa - \frac{r^2 R}{2} \right) u^3,$$

where $\kappa|_r$ is the Gauss curvature of $\gamma(r)$ and

$$\begin{aligned} A &= r \frac{\partial \bar{H}}{\partial r} - \bar{H} + \frac{1}{2} |\bar{\chi}|_\gamma^2 + \frac{1}{2} \bar{H}^2, \\ \bar{\chi}_{AB} &= \gamma_{AB} + \frac{1}{2} r \frac{\partial \gamma_{AB}}{\partial r}, \\ \bar{H} &= \text{tr}_\gamma \bar{\chi} = 2 + \frac{1}{2} r \frac{\partial \gamma_{AB}}{\partial r} \gamma^{AB}, \end{aligned}$$

A, B are used to denote components with respect to local coordinates (θ^1, θ^2) on \mathbb{S}^2 . For derivations of this equation see [1], [15], [13]. The equation has also been used successfully in [14] [18]. For a discussion of the use of this equation to construct non-time-symmetric initial data see [2], [3], [12].

Note that if $\bar{H} > 0$ and u is bounded above and below by positive constants then this equation is uniformly parabolic. As already noted, this is equivalent to the positivity of the mean curvature of the foliation spheres. Of course, this requires some pointwise a priori bounds on the solution u , but in the case that u is positive and bounded initially, these are easily obtained by the maximum principle under appropriate assumptions on the free data. The result of this is contained in the next lemma, whose proof can be found in [15], [16], and is a slight generalization of a result contained in [1]. The proof is repeated here since it yields simple but important bounds on the components of the constructed metric. To state this, given a function f on $[r_1, r_2] \times \mathbb{S}^2$ put $f^*(r) = \sup_{p \in \mathbb{S}^2} f(r, p)$, $f_*(r) = \inf_{p \in \mathbb{S}^2} f(r, p)$.

Lemma 1. *For $r \in I \equiv [r_1, r_2]$, let $\gamma(r)$ be a family of metrics on \mathbb{S}^2 and let \bar{H}, κ, A, R be defined in terms of γ as above. Define*

$$K_I = \sup_{r \in I} \left\{ \frac{1}{r_1} \int_{r_1}^r \left(\frac{2}{\bar{H}} \left(\frac{r^2 R}{2} - \kappa \right) \right)^* (s) \exp \left(\int_{r_1}^s \left(\frac{2A}{\bar{H}} - 1 \right)^* \frac{dt}{t} \right) ds \right\}.$$

Assume $\bar{H} > 0$ and $K < \infty$. Suppose that u is a solution of equation (2) on $[r_1, r_2]$ such that “initially” one has $0 < u(r_1, \cdot) < 1/\sqrt{K}$. Then there exists a constant C depending on γ, R, u_1, I such that $C^{-1} < u < C$.

Proof. In order to prove the result, it is useful to introduce the auxiliary function $w = u^{-2}$ to convert Equation (2) into an equation which is linear everywhere but the principal part:

$$r \partial_r w = -\frac{2}{\bar{H}} \frac{\Delta u}{u} - \frac{2A}{\bar{H}} w - \frac{2}{\bar{H}} \left(\frac{r^2 R}{2} - \kappa \right)$$

By using the maximum principle (see [16]) one can bound a solution of this equation with initial data w_1 from below by a solution of the ordinary differential equation

$$r \frac{dw_*}{dr} = - \left(\frac{2A}{\bar{H}} \right)^* w_* - \left(\frac{2}{\bar{H}} \left(\frac{r^2 R}{2} - \kappa \right) \right)^*$$

with any initial data satisfying $w_*(r_1) \leq w(r_1, \cdot)$. For convenience put $a = 2A/\bar{H}$, $b = r^2 R/2 - \kappa$ so that the previous equation can be written $rw' + a^*w = -b^*$. Using the integrating factor $\exp \int_{r_1}^r (a^* - 1)/t dt$ converts the equation into the form

$$\frac{d}{dr} \left(r \exp \int_{r_1}^r \left(\frac{a^* - 1}{t} \right) dt w_* \right) = -b^* \exp \int_{r_1}^r \left(\frac{a^* - 1}{t} \right) dt,$$

which can be immediately integrated to yield

$$w_* = \frac{1}{r} \exp \int_{r_1}^r \left(\frac{1 - a^*}{t} \right) dt \left\{ r_1 w_*(r_1) - \int_{r_1}^r b^* \exp \int_{r_1}^s \left(\frac{a^* - 1}{t} \right) dt ds \right\}.$$

The hypothesis of the lemma ensures that on the interval I the right hand side will be bounded below by a positive constant. This in turn gives the upper bound for u . To obtain the lower bound for u , we note that by using the maximum principle

again w can be bounded above by solutions of

$$r \frac{dw^*}{dr} = - \left(\frac{2A}{\bar{H}} \right)_* w^* - \left(\frac{2}{\bar{H}} \left(\frac{r^2 R}{2} - \kappa \right) \right)_*$$

satisfying $w^*(r_0) > w(r_0, \cdot)$. \square

If, in addition, the coefficients of Equation (2) are C^∞ then the bounds on u just obtained guarantee existence of a C^∞ solution with the initial data u_1 . This is carried out in [1], [15]. The result is contained in the next theorem.

Theorem 2. *Suppose that the hypotheses of the previous lemma are satisfied for $\gamma, R \in C^\infty([r_1, r_2] \times \mathbb{S}^2)$ and $u_1 \in C^\infty(\mathbb{S}^2)$. Then there exists a unique C^∞ solution u of Equation (2) on $I \equiv [r_1, r_2]$ with the initial data $u(r_1, \cdot) = u_1$. Furthermore $C^{-1} < u < C$, where C is the constant from the previous lemma.*

Note that in the previous theorem r_2 can be as large as we would like so that, in fact, we get existence as $r \rightarrow \infty$ as long as the free data continue to satisfy the hypotheses of the theorem on any interval. In our case we would actually like to obtain an asymptotically flat metric as a consequence of this. Hence, we need to assume appropriate asymptotic behavior on γ, R , which in our case we have done, and derive from this the correct asymptotic behavior of the solution u . As in previous work for Equation (2), this is done by proving the boundedness of the function m defined by

$$u^{-2} = 1 - \frac{2m}{r}.$$

The function m verifies the equation

$$\bar{H} r \partial_r m = \frac{\Delta_\gamma u}{u} - (2A - \bar{H}) m + r(A - B).$$

Since we have made the assumptions that on a neighborhood of infinity γ is identically the round metric and R vanishes, the equation simplifies dramatically. Indeed, we have $\bar{H} \equiv 2, A = 1$ so that the equation for m becomes

$$(3) \quad 2r \partial_r m = \frac{\Delta_\gamma u}{u}.$$

Bartnik has made a detailed analysis of this equation in [1], in which the following is proved:

Theorem 3. *Suppose that m is defined and C^∞ on $[r_0, \infty) \times \mathbb{S}^2$ and satisfies Equation (3), where m, u are related as above. Then there is a constant m_0 and $\varphi \in \mathbb{S}^2$ verifying $(\Delta + 2)\varphi = 0$ such that*

$$m = m_0 + \frac{\varphi}{r - 2m_0} + \epsilon(r),$$

where ϵ is a C^∞ function on $[r_0, \infty) \times \mathbb{S}^2$ satisfying,

$$|(r \partial_r)^i \nabla^j \epsilon| \leq C_{i,j} r^{-3} \log r,$$

for all $i, j \geq 0$ such that $i + j \leq k$.

This theorem shows, in particular, that the constructed metric g will be asymptotically flat in the sense mentioned in the introduction. Furthermore, the mass of the manifold is m_0 .

Putting the previous results together, we get the result needed to construct the metric outside of the collar region.

Theorem 4. *For $r \in [r_1, \infty)$ let $\gamma(r)$ be a family of metrics on \mathbb{S}^2 satisfying $\bar{H} > 0$ and $K < \infty$, where $K = K_{[r_0, \infty)}$ is defined as in Theorem 2. Furthermore, assume for some $r_2 > r_1$ that γ is the fixed round metric on \mathbb{S}^2 and $R \equiv 0$ for $r > r_2$. Then if $u_1 \in C^\infty(\mathbb{S}^2)$ satisfies $0 < u_1 < 1/\sqrt{K}$, there exists a unique C^∞ asymptotically flat metric g in the form*

$$g = u^2 dr^2 + r^2 \gamma,$$

with $u(r_1, \cdot) = u_1$, and whose scalar curvature satisfies $R(g) = R$.

3. CONSTRUCTING THE DATA

To construct the black hole initial data we construct the data on the collar region, and make sure this patches smoothly with data constructed via Theorem 4 for $r > r_1$. At first glance, it seems problematic to use Equation (2) to solve for u on the collar region since at $r = r_0$ the equation must fail to be parabolic if this surface is to be a horizon. This problem is resolved by assuming that γ is fixed on this region. Assuming in addition that v is also fixed in r on the collar region yields an elliptic equation for v on \mathbb{S}^2 . Indeed, the fact that γ does not vary in r immediately gives that $\bar{H} \equiv 2, A \equiv 1$ so that Equation (2) becomes

$$2r \frac{\partial u}{\partial r} = u^2 \Delta_\gamma u + u - \kappa u^3.$$

Substituting $u = v(1 - r_0/r)^{-\frac{1}{2}}$ yields the following equation for v :

$$(4) \quad \Delta_\gamma v - \kappa v + \frac{1}{v} = 0.$$

One has

Theorem 5. *Assume $\kappa > 0$ and let $\kappa_* = \inf_{\mathbb{S}^2} \kappa, \kappa^* = \sup_{\mathbb{S}^2} \kappa$. Then Equation (4) has a positive solution $v \in C^\infty$ satisfying $1/\sqrt{\kappa^*} \leq v \leq 1/\sqrt{\kappa_*}$. Within the class of C^2 functions there are exactly two solutions $\pm v$.*

Proof. We are going to produce the solution by the method of sub-solutions. Before starting the process, we note that sub(super)-solutions for Equation (4) in the sense of

$$\Delta_\gamma v - \kappa v + \frac{1}{v} \geq (\leq) 0$$

are also sub-solutions and super-solutions in the sense that any super-solution bounds any sub-solution from above. As usual, this is just a consequence of the maximum principle, but we should carefully check this anyway: Assuming that v_*, v^* are sub and super solutions of Equation (4) in the sense of the previous inequality, we consider the equation for the difference $\delta v^* = v^* - v_*$

$$\Delta \delta v^* - \left(\kappa + \frac{1}{v_* v^*} \right) \delta v^* \leq 0.$$

To see that $\delta v^* \geq 0$, suppose instead that $\delta v^* < 0$ somewhere, and consider the set $\Omega \equiv \{p \in \mathbb{S}^2 : \delta v^*(p) < 0\}$. Since the term in parentheses is positive, and since we have assumed Ω is nonempty, by the maximum principle one has $\inf_\Omega \delta v^* = \inf_{\partial\Omega} \delta v^*$, which is a contradiction since $\delta v^* = 0$ on $\partial\Omega$.

To now begin the method of sub-solutions, we are going to take as our starting sub-solution $v_* = 1/\sqrt{\kappa_*}$. For future reference, note that $v^* = 1/\sqrt{\kappa_*}$ is a super-solution. Choose $\lambda > 1/v_*^2$, and define v_1 as the solution of

$$\Delta v_1 - (\lambda + \kappa)v_1 = -\lambda v_* - \frac{1}{v_*}.$$

Then an application of the maximum principle just as in the first paragraph shows that $v_1 > v_*$ since the difference $\delta v_1 = v_1 - v_*$ satisfies

$$\Delta \delta v_1 - (\lambda + \kappa)\delta v_1 \leq 0.$$

We now define v_i inductively as the solutions of

$$(5) \quad \Delta v_i - (\lambda + \kappa)v_i = -\lambda v_{i-1} - \frac{1}{v_{i-1}}.$$

The fact that $v_i \geq v_{i-1}$ now follows from induction and another application of the maximum principle since $v_{i-1} \geq v_{i-2}$ yields for the difference $\delta v_i = v_i - v_{i-1}$ the inequality

$$\Delta \delta v_i - (\lambda + \kappa)\delta v_i = -\left(\lambda - \frac{1}{v_{i-1}v_{i-2}}\right)\delta v_{i-1} \leq 0.$$

Thus, we now have an increasing sequence of functions v_i , which are, in fact sub-solutions since for the right hand side of (5) we have

$$-\lambda v_{i-1} - \frac{1}{v_{i-1}} = -\lambda v_i - \frac{1}{v_i} + \left(\lambda - \frac{1}{v_i v_{i-1}}\right)(v_i - v_{i-1}) \geq -\lambda v_i - \frac{1}{v_i}.$$

Hence, we now have a pointwise non-decreasing sequence of sub-solutions v_i that satisfies $v_* \leq v_i \leq v^*$. Thus v_i converges to some function v pointwise. In addition, the bounds on v_i imply that the right hand side of the Equations (5) are uniformly bounded in L^q for any q . Whence, by applying elliptic regularity theory we see that the v_i are also uniformly bounded in $W^{2,q}$. Applying elliptic regularity iteratively, we may bootstrap to see that, in fact, the v_i are uniformly bounded in $W^{k,q}$ for any k, q so that the Sobolev embedding theorem shows that they are uniformly in C^k for any k . The Ascoli-Arzelà theorem then shows that a subsequence converges in C^k to a function that can be none other than v . Taking the limit in Equation (5) then shows that v must be a solution of Equation (4).

Concerning the uniqueness, first note that if a solution is to be of class C^2 , then it cannot change signs, and so we may restrict to the class of positive functions. Given two positive solutions v_1, v_2 one can see that $v_1 = v_2$ by, again, applying the maximum principle to the equation for the difference $\delta v = v_2 - v_1$

$$\Delta \delta v - \left(\kappa + \frac{1}{v_1 v_2}\right)\delta v = 0.$$

□

We may now prove our final theorem, of which the main theorem is an immediate corollary.

Theorem 6. *Let γ be a family of metrics as in the hypothesis of the main theorem with the exception that we do not assume $\kappa(\gamma) > 0$ outside of the collar region. Instead, assume that γ, ρ are such that, with $R = 16\pi\rho$ and K defined as in Theorem 4, one has*

$$K < r_0^2 \kappa(h) \left(1 - \frac{r_0}{r_1}\right).$$

Then there exists asymptotically flat time symmetric initial data $(g, 0)$ on M of mass density ρ such that g has the form

$$g = \frac{v^2}{1 - \frac{r_0}{r}} dr^2 + r^2 \gamma,$$

where v is C^∞ and bounded above and below by positive constants.

Proof. From the previous theorem we have the metric in the above form on the collar region. Taking $u_1 = v/\sqrt{1 - r_0/r_1}$ in Theorem 4 gives the metric for $r > r_1$ since $0 < u_1 < 1/\sqrt{K}$. We only have to check that the constructed metric is smooth across $r = r_1$. This is easily done by applying Theorem 4 again for $r \in [r_1 - \delta, \infty)$ for very small δ . Indeed, the new metric is certainly C^∞ at $r = r_1$ and agrees with the previous metric on $[r_1 - \delta, \infty)$ by the uniqueness part of that theorem. \square

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